

# Classical Mechanics and Geometry

## 经典力学与几何

(preliminary draft updated July 2023)



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You can also contact me at [sili@mail.tsinghua.edu.cn](mailto:sili@mail.tsinghua.edu.cn). The draft will be updated on my homepage: <https://sili-math.github.io/>. Thank you.

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## Preface

In April 2021, Qiu Zhen College (求真书院) was newly established at Tsinghua University under the leadership of Professor Shing-Tung Yau. It homes the distinguished elite mathematics program in China starting in 2021: the “Yau Mathematical Sciences Leaders Program” (丘成桐数学科学领军人才培养计划). This program puts strong emphasis on basic sciences related to mathematics in a broad sense. Though majored in mathematics, students in this program are required to study fundamental theoretical physics such as classical mechanics, electromagnetism, quantum mechanics, and statistical mechanics, in order to understand global perspectives of theoretical sciences. It is an exciting challenge both for students and for instructors.

This preliminary note is written for the course “Classical Mechanics” that I lectured at Qiu Zhen College in the fall semester of 2022. It is to explain key physics ingredients of Lagrangian and Hamiltonian mechanics, as well as their connections with modern geometric development. We put heavy emphasis on different faces of concrete examples in order to understand the bridge between mathematics and physics. Examples such as Toda lattice and Calogero-Moser System are still active research topics nowadays in areas of integrable system, representation theory and mathematical physics. A large part of this note relies on the beautiful books of “Mechanics” by Landau-Lifshitz, and “Mathematical Methods of Classical Mechanics” by Arnol’d, which themselves show different faces of this classical subject. Other useful resources that we consulted are listed at the end of this note.

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静斋

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